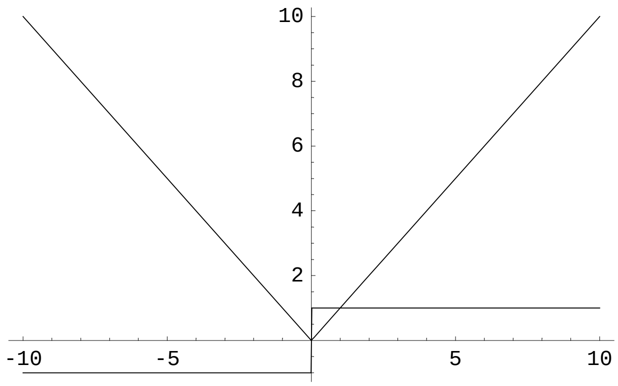
```
(* 2-parameter ICE preference maps in Cartesian Coordinates *)

Plot[{Abs[x], Sign[x]}, {x, -10, 10}]
```




```
- Graphics -

(* Obscure discontinuities in Sign function and derivative of Abs function at x = 0 *)

Sign'[x_] := 0;

Abs'[x_] := Sign[x];

(* 2-Parameter ICE Preference Maps: Functional DEFINITION *)
```

P[x_, y_, β_, γ_] := $(x^2 + y^2)^{(\beta-\gamma)/2}$ Sign[x - y] Abs[x - y]$^\gamma$;

Derivative[1, 0, 0, 0][P][x, y, β, γ]

$x\,(x^2+y^2)^{-1+\frac{\beta-\gamma}{2}}\,(\beta-\gamma)\,\mathrm{Abs}[x-y]^{\gamma}\,\mathrm{Sign}[x-y] + (x^2+y^2)^{\frac{\beta-\gamma}{2}}\,\gamma\,\mathrm{Abs}[x-y]^{-1+\gamma}\,\mathrm{Sign}[x-y]^2$

(* NOTE: Sign[]^2 is almost always 1,
 but this cannot be declared because "Tag Power is Protected." *)

FullSimplify[Derivative[1, 0, 0, 0][P][x, y, β, γ]]

$(x^2+y^2)^{\frac{1}{2}(-2+\beta-\gamma)}\,\mathrm{Abs}[x-y]^{-1+\gamma}\,\mathrm{Sign}[x-y]\,(x\,(\beta-\gamma)\,\mathrm{Abs}[x-y] + (x^2+y^2)\,\gamma\,\mathrm{Sign}[x-y])$

(* Now, define dPdx[] = derivative of P[] with respect to x *)

dPdx[x_, y_, β_, γ_] := + $(x^2 + y^2)^{-1+(\beta-\gamma)/2}$ Abs[x - y]$^{-1+\gamma}$ $(\beta\, x^2 + (\gamma - \beta)\, xy + \gamma\, y^2)$;

Derivative[0, 1, 0, 0][P][x, y, β, γ]

$y\,(x^2+y^2)^{-1+\frac{\beta-\gamma}{2}}\,(\beta-\gamma)\,\mathrm{Abs}[x-y]^{\gamma}\,\mathrm{Sign}[x-y] - (x^2+y^2)^{\frac{\beta-\gamma}{2}}\,\gamma\,\mathrm{Abs}[x-y]^{-1+\gamma}\,\mathrm{Sign}[x-y]^2$

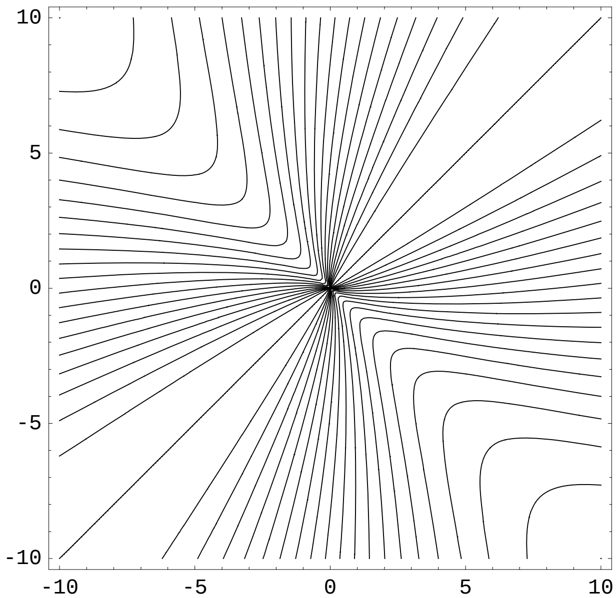
```
FullSimplify[Derivative[0, 1, 0, 0][P][x, y, β, γ]]
```

$-(x^2+y^2)^{\frac{1}{2}(-2+\beta-\gamma)}\,\mathrm{Abs}[x-y]^{-1+\gamma}\,\mathrm{Sign}[x-y]\,(y\,(-\beta+\gamma)\,\mathrm{Abs}[x-y]+(x^2+y^2)\,\gamma\,\mathrm{Sign}[x-y])$

```
(* Next, define dPdy[] = derivative of P[] with respect to y *)
```

$\mathrm{dPdy}[x\_, y\_, \beta\_, \gamma\_] := -(x^2+y^2)^{-1+(\beta-\gamma)/2}\,\mathrm{Abs}[x-y]^{-1+\gamma}\,(\gamma\,x^2+(\gamma-\beta)\,x\,y+\beta\,y^2);$

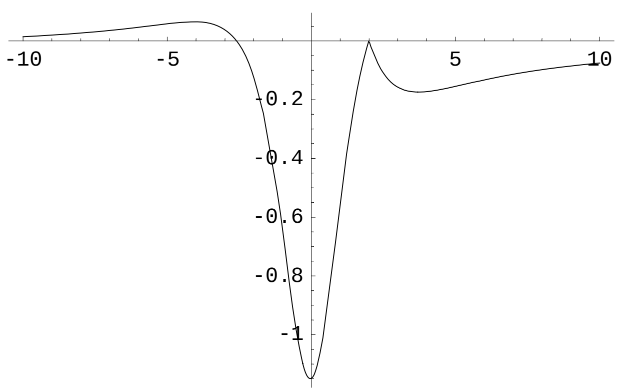
```
ContourPlot[P[x, y, 0.2, 2], {x, -10, 10}, {y, -10, 10},
 PlotPoints → 250, Contours → 41, ContourShading → False]
```



- ContourGraphics -

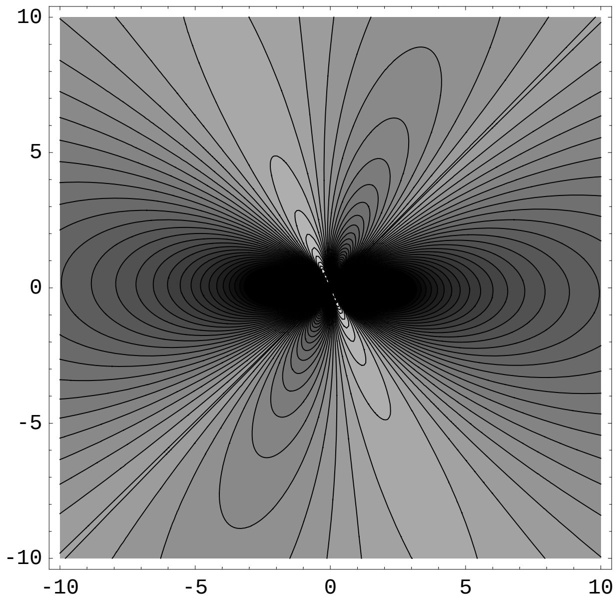
```
(* But the following dPdy derivative
   is NOT always negative. NOTE: γ/β = 10 is large here *)
```

```
Plot[dPdy[2, y, 0.2, 2], {y, -10, 10}]
```



- Graphics -

```
ContourPlot[dPdy[x, y, 0.2, 2], {x, -10, 10},
 {y, -10, 10}, PlotPoints → 250, Contours → 41]
```



- ContourGraphics -

(* When can the $(\gamma\ x^2+(\gamma-\beta)\ x\ y+\beta\ y^2)$ factor in dPdy[] change sign? *)

Solve[$\gamma\, x^2 + (\gamma - \beta)\, x\, y + \beta\, y^2 == 0$, x]

$$\left\{\left\{x \to \frac{-y\,(-\beta+\gamma) - y\sqrt{\beta^2 - 6\,\beta\,\gamma + \gamma^2}}{2\,\gamma}\right\}, \left\{x \to \frac{-y\,(-\beta+\gamma) + y\sqrt{\beta^2 - 6\,\beta\,\gamma + \gamma^2}}{2\,\gamma}\right\}\right\}$$

(* When will $\gamma^2-6\,\gamma\,\beta + \beta^2$ be negative? ...i.e. no real solution to above equation! *)

Solve[$\gamma^2 - 6\,\gamma\,\beta + \beta^2 == 0$, γ]

$$\left\{\left\{\gamma \to 3\,\beta - 2\sqrt{2}\,\beta\right\}, \left\{\gamma \to 3\,\beta + 2\sqrt{2}\,\beta\right\}\right\}$$

(* No real solution when γ/β is less than the upper bound: $3+2\sqrt{2} = \left(1+\sqrt{2}\right)^2$. *)

N[$3 + 2\sqrt{2}$, 25]

5.828427124746190097603377

(* Note that lower bound of $3-2\sqrt{2}$ is also the reciprocal of the upper bound. *)

```
N[3 - 2 √2, 25]
```

0.1715728752538099023966226

```
(* This notebook explores properties of the most "directional" ICE preference *)
(* maps that satisfy the axiom of Cartesian Monotonicity [CM]. *)
(*    These extreme maps are called "ICE-Omega" preference maps because their *)
(* power-parameter-ratio, γ/β, equals the CM upper limit of Ω = (1 + √2)^2, *)
(* which is approximately 5.828 *)
```

$$P[x\_, y\_, \beta\_, \Omega\_] := (x^2 + y^2)^{\beta (1-\Omega)/2}\, \mathrm{Sign}[x - y]\, \mathrm{Abs}[x - y]^{\beta \Omega};$$

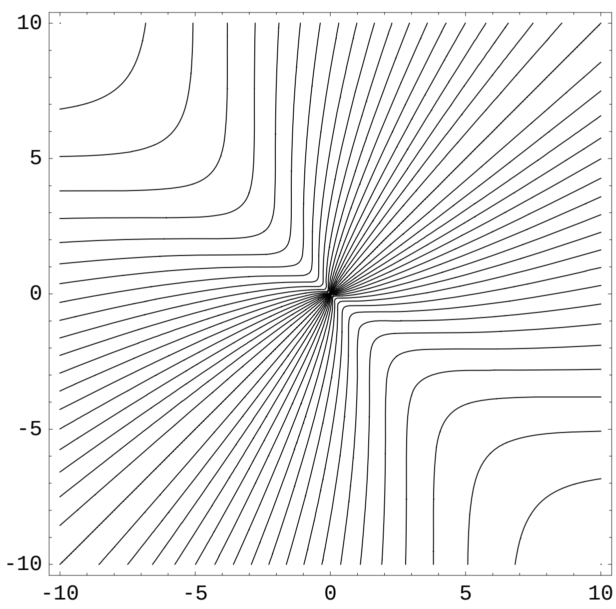
```
(* Note that {1-Ω}/2 = -√Ω = -1 - √2. *)
```

$$\mathrm{FullSimplify}\left[\left\{1 - \left(1 + \sqrt{2}\right)^2\right\} / 2\right]$$

$$\left\{-1 - \sqrt{2}\right\}$$

```
(* The ContourPlot below depicts an ICE-Omega map which, due to β = 0.2 < 1, *)
(* corresponds to decreasing returns-to-sacle. *)
```

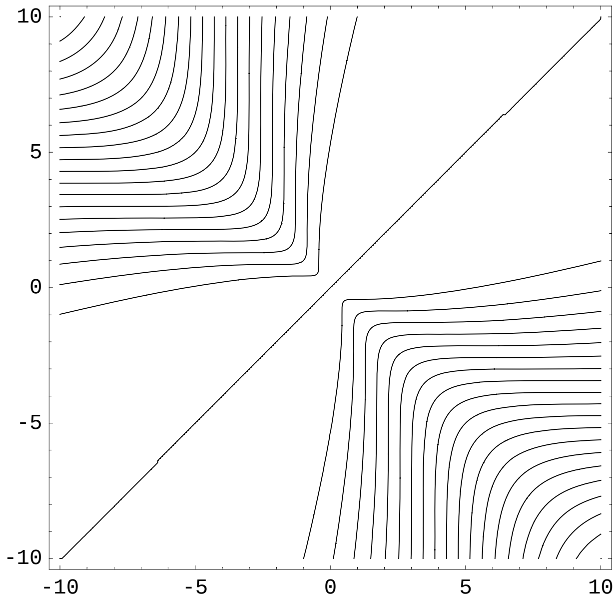
$$\mathrm{ContourPlot}\left[P\left[x, y, 0.2, \left(1 + \sqrt{2}\right)^2\right], \{x, -10, 10\}, \{y, -10, 10\},\right.$$
$$\left.\mathrm{PlotPoints} \to 250,\ \mathrm{Contours} \to 41,\ \mathrm{ContourShading} \to \mathrm{False}\right]$$



- ContourGraphics -

```
(* Next, here is the ICE-Omega map with linear [constant] returns-to-sacle, β = 1. *)
```

```
ContourPlot[P[x, y, 1, (1 + √2)^2], {x, -10, 10}, {y, -10, 10},
 PlotPoints → 250, Contours → 41, ContourShading → False]
```




```
- ContourGraphics -
```

```
(* Willingness is a function of only the std. ICE ratio ρ = x/y when γ/β == Ω. *)

Willingness[ρ_, Ω_] := (1 + (Ω - 1) ρ + Ω ρ^2) / (Ω + (Ω - 1) ρ + ρ^2);

Solve[Willingness[ρ, (1 + √2)^2] == 0, ρ]
```

$\{\{\rho \to 1 - \sqrt{2}\}, \{\rho \to 1 - \sqrt{2}\}\}$

```
Solve[Derivative[1, 0][Willingness][ρ, (1 + √2)^2] == 0, ρ]
```

```
Solve::verif : Potential solution {ρ → -1 - √2} (possibly
   discarded by verifier) should be checked by hand. May require use of limits. More…
```

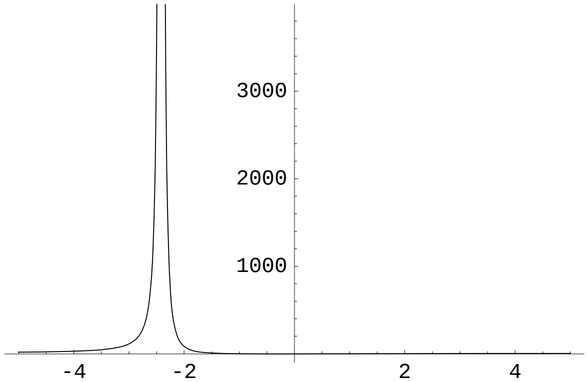
$\{\{\rho \to 1 - \sqrt{2}\}\}$

```
N[Solve[Derivative[1, 0][Willingness][ρ, (1 + √2)^2] == 0, ρ]]
```

```
Solve::verif : Potential solution {ρ → -1 - √2} (possibly
   discarded by verifier) should be checked by hand. May require use of limits. More…
```

$\{\{\rho \to -0.414214\}\}$

```
Plot[{Willingness[ρ, (1 + √2)^2]}, {ρ, -5, 5}]
```




```
- Graphics -

N[Willingness[-1 - √2, (1 + √2)^2]]

Power::infy : Infinite expression 1/0. encountered. More…

Power::infy : Infinite expression 1/0.^1. encountered. More…

ComplexInfinity

N[Willingness[1 - √2, (1 + √2)^2]]

0.

(*    Unlike other 2 parameter ICE preference maps possessing Monotonicity, *)
(* Willingness within ICE-Omega maps ranges from 0 to +∞.        In maps with a *)
(* power-parameter-ratio = γ/β that is strictly between 1/Ω and Ω, the minimum *)
(* Willingness is strictly positive, and the maximum Willingness is bounded. *)

Solve[Derivative[1, 0][Willingness][ρ, η] == 0, ρ]
```

$$\left\{\left\{\rho \to \frac{-1 - 2\sqrt{\eta} - \eta}{-1 + \eta}\right\}, \left\{\rho \to \frac{-1 + 2\sqrt{\eta} - \eta}{-1 + \eta}\right\}\right\}$$

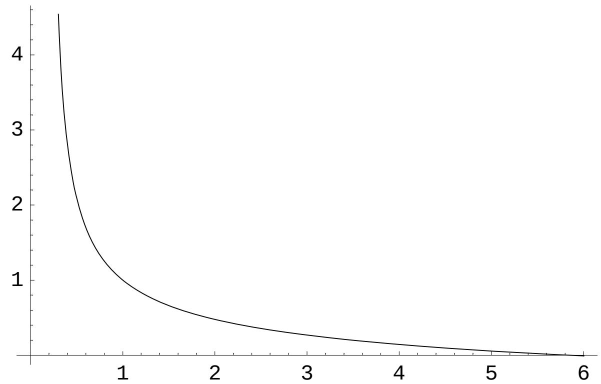
```
(* Note that the above roots are reciprocals of each other ...
  both have the same numerical sign. *)
```

$$\text{FullSimplify}\left[\text{Willingness}\left[\frac{1 - \sqrt{\eta}}{1 + \sqrt{\eta}}, \eta\right]\right]$$

$$-1 + \frac{4\sqrt{\eta}}{-1 + 2\sqrt{\eta} + \eta}$$

```
Plot[-1 + 4 Sqrt[η]/(-1 + 2 Sqrt[η] + η), {η, 0.3, 6}]
```

$$\text{Plot}\left[-1+\frac{4\sqrt{\eta}}{-1+2\sqrt{\eta}+\eta},\ \{\eta,\ 0.3,\ 6\}\right]$$

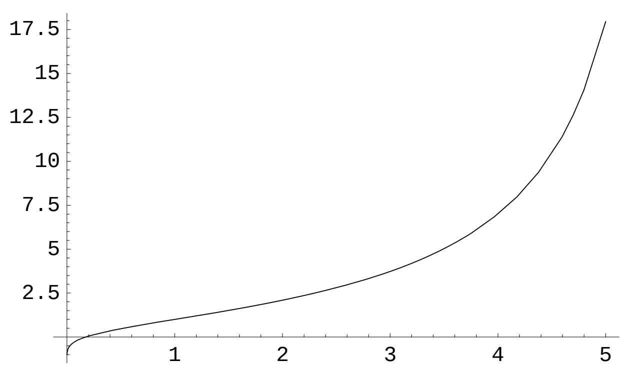


- Graphics -

$$\text{FullSimplify}\left[\text{Willingness}\left[\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}},\ \eta\right]\right]$$

$$-1-\frac{4\sqrt{\eta}}{-1-2\sqrt{\eta}+\eta}$$

$$\text{Plot}\left[-1+\frac{4\sqrt{\eta}}{1+2\sqrt{\eta}-\eta},\ \{\eta,\ 0,\ 5\}\right]$$



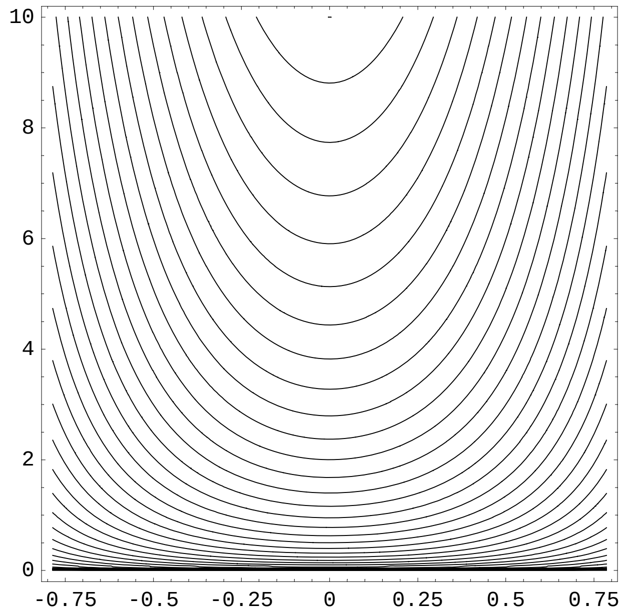
- Graphics -

```
(* Visualization of ICE-Omega preference maps in polar coordinates. *)
```

Ψ[r_, θ_, β_, Ω_] := r^{β} Sign[Cos[θ]] Abs[Cos[θ]]$^{\beta\,\Omega}$;

```
(* The graph below depicts ICE-Omega preferences within the South-East Quadrant. *)
```

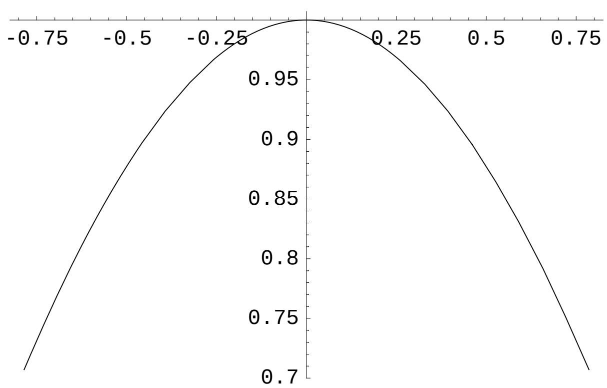
```
ContourPlot[Ψ[r, θ, 0.2, (1 + √2)^2], {θ, -π / 4, π / 4},
 {r, 0, 10}, PlotPoints → 250, Contours → 41, ContourShading → False]
```



- ContourGraphics -

```
(* When ICE radius is restricted, such as "r at most 10" in the above graph, some *)
(* relatively large values of ICE preference can be achived within the SE quadrant *)
(* that cannot be achieved within any other quadrant. *)
```

```
Plot[Cos[θ], {θ, -π / 4, π / 4}]
```



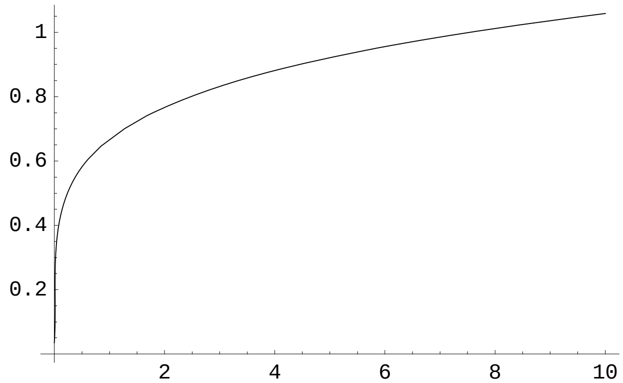
- Graphics -

```
(* Along the θ = π/4 boundary between the SE and NE quadrants, preference within *)
(* ICE-Omega maps can be made arbitrarily large simply by increasing the ICE *)
(* radius, r, as long as one restricts β to be strictly positive. *)
```

```
FullSimplify[Ψ[r, π / 4, β, (1 + √2)^2]]
```

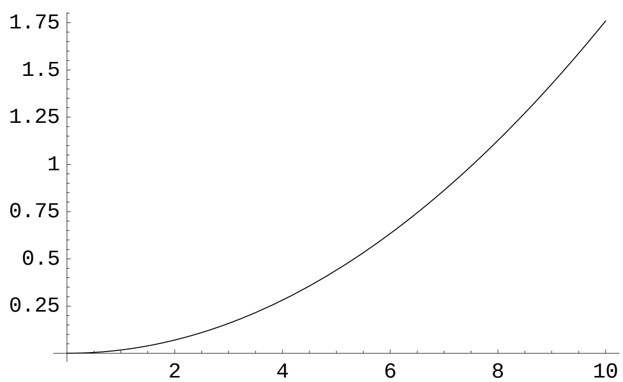
$2^{-\frac{1}{2}(3+2\sqrt{2})\beta}\, r^{\beta}$

```
Plot[Ψ[r, π / 4, 0.2, (1 + √2)^2], {r, 0, 10}]
```



- Graphics -

```
Plot[Ψ[r, π / 4, 2, (1 + √2)^2], {r, 0, 10}]
```



- Graphics -

```
Will2[θ_, Ω_] := (1 + (Ω - 1) Tan[θ - π / 4] + Ω Tan[θ - π / 4]^2) /
    (Ω + (Ω - 1) Tan[θ - π / 4] + Tan[θ - π / 4]^2);
```

```
Solve[Will2[θ, (1 + √2)^2] == 0, θ]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. More…

$\left\{\left\{\theta \to \frac{1}{4}\left(\pi + 4\,\mathrm{ArcTan}\left[1 - \sqrt{2}\right]\right)\right\}\right\}$

```
(* Angle in radians. *)
```

```
N[1/4 (π + 4 ArcTan[1 - √2])]
```

0.392699

```
(* Angle in degrees. *)

N[180 * 0.392699 / π]
```

22.5

```
(* Slope of Ray. *)

N[Sin[1/4 (π + 4 ArcTan[1 - √2])]]
```

0.382683

```
N[1 / Sin[1/4 (π + 4 ArcTan[1 - √2])]]
```

2.61313

```
(* Willingness is ZERO at θ = π/8 = +22.5 degrees. *)

N[Will2[π / 8, (1 + √2)^2]]
```

5.55112×10^{-17}

```
(* Willingness is +∞ at θ = -π/8 = -22.5 degrees. *)

N[Will2[-π / 8, (1 + √2)^2]]
```

Power::infy : Infinite expression $\frac{1}{0.}$ encountered. More…

Power::infy : Infinite expression $\frac{1}{0.^{1.}}$ encountered. More…

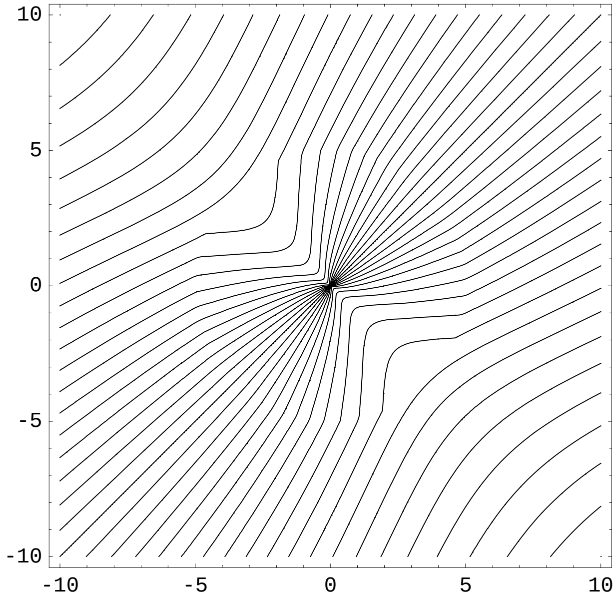
ComplexInfinity

```
(* Visualizing 4-parameter ICE preference maps... *)

P[x_, y_, β_, η_, γ_, ρ_] := If[x^2 + y^2 < ρ^2,
   ((x^2 + y^2) / ρ^2)^((β-γ)/2) Sign[x - y] Abs[x - y]^γ, ((x^2 + y^2) / ρ^2)^((η-γ)/2) Sign[x - y] Abs[x - y]^γ];

(* Inside the circle of radius ρ,
 one rescaling of the 2-parameter (β,γ) map applies. *)
(* Outside the circle of radius ρ,
 a different rescaling of the 2-parameter (η,γ) map applies. *)
(* On the circle of radius ρ,
 differences between β and η are unimportant because 1 to any power is 1. *)
```

```
ContourPlot[P[x, y, 0.2, 0.5, 1, 5], {x, -10, 10},
 {y, -10, 10}, PlotPoints → 250, Contours → 41, ContourShading → False]
```



- ContourGraphics -